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A class of integro-differential equations constrained from the KP hierarchy

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Abstract. In this paper we give a class of integro-differential evolution equations which generalizes the ILW_k hierarchy. We show the generalized Lax pairs, the bilinear representations and the rational and N-soliton solutions for these integro-differential equations. We also give the Hamiltonian formalism for the simplest set of equations among these integro-differential equations, which is the coupling of the ILW equation and the time-dependent Schrödinger equations.

1. Introduction

We know that many well known integrable systems in $1+1$ dimensions, such as the KdV and Boussinesq equations, can be reduced from the following $(2+1)$-dimensional KP hierarchy [1]:

$$P_{t_n} = -(P\partial^n P^{-1})_- P \qquad n = 1, 2, \ldots \tag{1.1}$$

where

$$P = 1 + w_1\partial^{-1} + w_2\partial^{-2} + \cdots \tag{1.2}$$

is a micro-differential operator with coefficients w_i ($i \geqslant 1$), depending on variables $t = (t_1, t_2, \ldots,)$, and $\partial = \frac{\partial}{\partial x}$ with $x = t_1$ and, if we denote the differential part of the micro-differential operator $P\partial^n P^{-1}$ by $(P\partial^n P^{-1})_+$, then $(P\partial^n P^{-1})_- = P\partial^n P^{-1} - (P\partial^n P^{-1})_+$. The reduction procedure is to impose on the micro-differential operator P the condition that $P\partial^k P^{-1}$ is a differential operator for certain positive integers k [1]. This reduction procedure was generalized in two ways in [2–6]. One way is to impose on P the constraint that, for certain positive integer k, P satisfies [4]

$$P\partial^k P^{-1} = B_k + q\partial^{-1} r \tag{1.3}$$

where $B_k = (P\partial^k P^{-1})_+$, q and r satisfy

$$q_{t_n} = B_n q \qquad r_{t_n} = -B_n^* r \qquad n = 1, 2, \ldots \tag{1.4}$$

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and $\partial^{-1}r$ is defined by

$$\partial^{-1}r = r\partial^{-1} - r_x\partial^{-2} + r_{xx}\partial^{-3} + \cdots . \tag{1.5}$$

B_n^* in (1.4) is the conjugate operator of B_n. One finds that this reduction procedure leads to the well known AKNS hierarchy when $k = 1$, and to the Yajima–Oikawa hierarchy [4] when $k = 2$, and so on. For general positive integer k, the hierarchy of equations obtained by the above reduction procedure is called the k-constrained KP hierarchy [4]. It has been shown that the k-constrained KP hierarchy has Lax pairs, recursion operators, bi-Hamiltonian structures in [4,7], and bilinear representations in [8].

Another way of generalizing the original reduction procedure of the KP hierarchy is given in [5,6]. In this case, the constraint condition imposed on the micro-differential operator P is given by

$$(\widetilde{P}\partial^k P^{-1})_- = 0 \tag{1.6}$$

where $\widetilde{P}$ is defined by

$$\widetilde{P} = 1 + \widetilde{w}_1\partial^{-1} + \widetilde{w}_2\partial^{-2} + \cdots \qquad \widetilde{w}_i(t_1, t_2, \cdots) = w(t_1 - 2i\hbar, t_2, \cdots) \tag{1.7}$$

and $\hbar$ is a constant. This reduction procedure leads to the ILW$_k$ hierarchy, which is the generalization of the well known ILW equation. It was shown in [5,9] that ILW$_k$ hierarchy possesses Hamiltonian structure and zero curvature representations.

In this paper, we will combine the above-mentioned approaches of reducing the KP hierarchy to obtain a new class of integro-differential evolution equations which generalize the ILW$_k$ hierarchy. The simplest set of equations we have obtained is the ILW equation coupled with the time-dependent Schrödinger equations. We shall also give the generalized Lax pairs and the bilinear representations for these integro-differential equations and then, by making use of the bilinear representations, we obtain their rational and N-soliton solutions. In the conclusion, we also remark on the Hamiltonian structure and the further generalization of these integro-differential equations.

2. The integro-differential equations

Let the micro-differential operators P, $\widetilde{P}$ be defined by (1.2) and (1.7), with P also satisfying equation (1.1). Instead of imposing the constraint of conditions (1.3) or (1.6) on P, we now impose the following constraint on P:

$$(\widetilde{P}\partial^k P^{-1})_- = q\partial^{-1}r \tag{2.1}$$

where k is any positive integer. This constraint leads to the following hierarchy of equations:

$$L_{t_n} = \widetilde{B}_n L - L B_n \tag{2.2a}$$

$$q_{t_n} = \widetilde{B}_n q \tag{2.2b}$$

$$r_{t_n} = -B_n^* r \qquad n = 1, 2, \ldots \tag{2.2c}$$

where $L = \widetilde{P}\partial^k P^{-1}$, $\widetilde{B}_n = (\widetilde{P}\partial^n \widetilde{P}^{-1})_+$, $B_n = (P\partial^n P^{-1})_+$ and B_n^* is the conjugate operator of B_n.

Since from constraint condition (2.1), we know that

$$\begin{aligned}(\widetilde{B}_n L - LB_n)_- &= (\widetilde{B}_n(\widetilde{P}\partial^k P^{-1})_- - (\widetilde{P}\partial^k P^{-1})_- B_n)_- \\ &= (\widetilde{B}_n q\partial^{-1} r - q\partial^{-1} r B_n)_- \\ &= (\widetilde{B}_n q)\partial^{-1} r + q\partial^{-1}(-B_n^* r)\end{aligned} \tag{2.3}$$

where $(\widetilde{B}_n q)$ and $(-B_n^* r)$ are scalar functions, we see that the hierarchy of equations (2.2) is well defined and compatible with constraint condition (2.1).

Let us express L as follows

$$L = \widetilde{P}\partial^k P^{-1} = \partial^k + u_1\partial^{k-1} + u_2\partial^{k-2} + \cdots + u_k + q\partial^{-1} r \tag{2.4}$$

then we see that, for a fixed positive integer n, equation (2.2) turns out to be a set of $(k+2)$ equations for $(k+2)$ unknown functions $u_1, u_2, \cdots, u_k, q, r$. So, equations in (2.2) form a hierarchy of nonlinear evolution equations in $1+1$ dimensions, and they are also integro-differential equations. For example, let us assume $k = 1$, $n = 2$, then we obtain the following set of equations:

$$q_{t_2} = q_{xx} - u_{1,x} q + \mathrm{i} q T u_{1,x} \tag{2.5a}$$

$$u_{1,t_2} = 2(qr)_x - 2u_1 u_{1,x} - \mathrm{i} T u_{1,xx} \tag{2.5b}$$

$$r_{t_2} = -r_{xx} - u_{1,x} r - \mathrm{i} r T u_{1,x} \tag{2.5c}$$

where the operator T is defined by [5]

$$Tf(x) = \frac{1}{2\hbar}\mathrm{P.V.}\int_{-\infty}^{\infty} \coth\left(\frac{\pi(y-x)}{2\hbar}\right) f(y)\,\mathrm{d}x \tag{2.6}$$

and P.V. indicates that the above integral is the principal value integral. Equation (2.5b), without the term $2(qr)_x$, is simply the ILW equation, which describes the propagation of internal long waves in a fluid of finite depth characterized by parameter $\hbar$ (see [6, 9] and references therein), while equations (2.5a) and (2.5c) are the time-dependent Schrödinger equations. Thus, we obtain equation (2.5), which is the coupling of the ILW equation with the time-dependent Schrödinger equations.

We note that when $q = r = 0$, the hierarchy of equations (2.2) reduces to the ILW_k hierarchy [5]. So, the hierarchy of equations (2.2) generalizes the ILW_k hierarchy.

We now define the wavefunctions ψ, $\widetilde{\psi}$ and their conjugates ψ^*, $\widetilde{\psi}^*$ for the hierarchy of equations given by (2.2), as follows

$$\psi(t,\lambda) = P\mathrm{e}^{\xi(t,\lambda)} \qquad \psi^*(t,\lambda) = (P^{-1})^*\mathrm{e}^{-\xi(t,\lambda)} \tag{2.7a}$$

$$\widetilde{\psi}(t,\lambda) = \widetilde{P}\mathrm{e}^{\xi(t,\lambda)} \qquad \widetilde{\psi}^*(t,\lambda)) = (\widetilde{P}^{-1})^*\mathrm{e}^{-\xi(t,\lambda)} \tag{2.7b}$$

$$\xi(t,\lambda) = \sum_{i=1}^{\infty} t_i\lambda^i \tag{2.7c}$$

then these wavefunctions satisfy the following linear equations:

$$(\widetilde{P}\partial^k P^{-1})_+\psi + q\varphi = \lambda^k\widetilde{\psi} \qquad \varphi_x = r\psi \tag{2.8a}$$

$$\psi_{t_n} = B_n\psi \qquad \widetilde{\psi}_{t_n} = \widetilde{B}_n\widetilde{\psi} \qquad \varphi_{t_n} = A_n\psi \tag{2.8b}$$

$$n = 1, 2, \ldots$$

and their conjugate equations

$$(\widetilde{P}\partial^k P^{-1})_+^* \widetilde{\psi}^* - r\widetilde{\varphi}^* = \lambda^k \psi^* \qquad \widetilde{\varphi}_x^* = q\widetilde{\psi}^* \tag{2.9a}$$

$$\psi_{t_n}^* = -B_n^* \psi^* \qquad \widetilde{\psi}_{t_n}^* = -\widetilde{B}_n^* \widetilde{\psi}^* \qquad \widetilde{\varphi}_{t_n}^* = \widetilde{A}_n^* \widetilde{\psi}^* \tag{2.9b}$$

$$n = 1, 2, \ldots$$

where the operators A_n and $\widetilde{A}_n^*$ are defined by

$$\partial A_n = rB_n - (B_n^* r) \qquad \partial \widetilde{A}_n^* = (\widetilde{B}_n q) - q\widetilde{B}_n^* \tag{2.10}$$

and $(B_n^* r)$, $(\widetilde{B}_n q)$ are scalar functions. It can be verified that equations (2.2) are just the compatibility conditions of equations (2.8) or (2.9), so equations (2.8) give the generalized Lax pairs for the hierarchy of equations (2.2). We note that equations (2.8) are similar to the Lax pair for the k-constrained KP hierarchy given in [4].

3. The bilinear equations for the hierarchy of equations (2.2)

We know that the KP hierarchy (1.1) corresponding to the operator P has the following bilinear representation [1]:

$$\mathrm{Res}_\lambda(\psi(t, \lambda)\psi^*(t', \lambda)) = 0 \tag{3.1}$$

and for the KP hierarchy corresponding to the operator $\widetilde{P}$, we have

$$\mathrm{Res}_\lambda(\widetilde{\psi}(t, \lambda)\widetilde{\psi}^*(t', \lambda)) = 0 \tag{3.2}$$

where t, t' are independent variables. To obtain the bilinear representation for the hierarchy of equations (2.2), we first need to express constraint condition (2.1) in bilinear form. By using a similar argument as the one given in [8] for deriving the bilinear representation of the k-constrained KP hierarchy, we know that condition (2.1) is equivalent to the following equation:

$$\mathrm{Res}_\lambda\{\lambda^k \widetilde{\psi}(t, \lambda)\psi^*(t', \lambda)\} = q(t)r(t'). \tag{3.3}$$

From equations (2.8a), (2.9a) and (3.1)–(3.3), we have

$$q(t) = -\mathrm{Res}_\lambda\{\widetilde{\psi}(t, \lambda)\widetilde{\varphi}^*(t', \lambda)\} \tag{3.4a}$$

$$r(t') = \mathrm{Res}_\lambda\{\varphi(t, \lambda)\psi^*(t', \lambda)\}. \tag{3.4b}$$

By using equations (3.1) and (3.2), we can show, as in the case of k-constrained KP hierarchy [8], that equations (3.3), (3.4a) and (3.4b) are equivalent to the hierarchy of equations (2.2).

The τ-function of the KP hierarchy can be defined by the following relations:

$$\psi(t, \lambda) = \frac{\tau(t - \epsilon(\lambda))}{\tau(t)} e^{\xi(t,\lambda)} \tag{3.5a}$$

$$\psi^*(t, \lambda) = \frac{\tau(t + \epsilon(\lambda))}{\tau(t)} e^{-\xi(t,\lambda)} \tag{3.5b}$$

where

$$\epsilon(\lambda) = \left(\frac{1}{\lambda}, \frac{1}{2\lambda^2}, \frac{1}{3\lambda^3}, \dots\right) \tag{3.6}$$

and $\xi(t, \lambda)$ is defined by (2.7*c*). The τ-function corresponding to operator $\widetilde{P}$ is denoted by $\widetilde{\tau}$ and has the following relation with τ:

$$\widetilde{\tau}(t_1, t_2, t_3, \dots) = \tau(t_1 - 2\mathrm{i}\hbar, t_2, t_3, \dots). \tag{3.7}$$

From relations (2.7) and (3.5)–(3.7), we can express the coefficients $u_i (i = 1, 2, \dots, k)$ of the operator L given by (2.4) in terms of the τ-functions τ and $\widetilde{\tau}$. We further express $q(t)$ and $r(t)$ as follows

$$q(t) = \frac{\rho(t)}{\widetilde{\tau}(t)} \qquad r(t) = \frac{\sigma(t)}{\tau(t)}. \tag{3.8}$$

Then, by using (3.4) and (3.5), we can express the wavefunctions φ and $\widetilde{\varphi}^*$ defined in (2.8) and (2.9) in the following form:

$$\varphi(t, \lambda) = \frac{\sigma(t - \epsilon(\lambda))}{\lambda\tau(t)} \mathrm{e}^{\xi(t,\lambda)} \tag{3.9a}$$

$$\widetilde{\varphi}^*(t, \lambda) = -\frac{\rho(t + \epsilon(\lambda))}{\lambda\widetilde{\tau}(t)} \mathrm{e}^{-\xi(t,\lambda)}. \tag{3.9b}$$

Expression (3.9) can be proved by using an argument similar to that given in [8]

Now, by substituting expressions (3.5), (3.8) and (3.9) into (3.3) and (3.4), we obtain the following proposition.

Proposition 3.1. The hierarchy of equations (2.2) has the following bilinear representation:

$$\mathrm{Res}_\lambda \left(\lambda^k \widetilde{\tau}(t - \epsilon(\lambda))\tau(t' + \epsilon(\lambda))\mathrm{e}^{\xi(t-t',\lambda)}\right) = \rho(t)\sigma(t') \tag{3.10a}$$

$$\mathrm{Res}_\lambda \left(\lambda^{-1} \sigma(t - \epsilon(\lambda))\tau(t' + \epsilon(\lambda))\mathrm{e}^{\xi(t-t',\lambda)}\right) = \sigma(t')\tau(t) \tag{3.10b}$$

$$\mathrm{Res}_\lambda \left(\lambda^{-1} \widetilde{\tau}(t - \epsilon(\lambda))\rho(t' + \epsilon(\lambda))\mathrm{e}^{\xi(t-t',\lambda)}\right) = \rho(t)\widetilde{\tau}(t'). \tag{3.10c}$$

The equations in (3.10) can be expressed more explicitly as follows

$$\sum_{j=0}^{\infty} p_j(-2y)p_{j+k+1}(\tilde{D}) \exp\left(\sum_{l=1}^{\infty} y_l D_l\right) \tau \cdot \widetilde{\tau} = \exp\left(\sum_{l=1}^{\infty} y_l D_l\right) \sigma \cdot \rho \tag{3.11a}$$

$$\sum_{j=0}^{\infty} p_j(-2y)p_j(\tilde{D}) \exp\left(\sum_{l=1}^{\infty} y_l D_l\right) \tau \cdot \sigma = \exp\left(\sum_{l=1}^{\infty} y_l D_l\right) \sigma \cdot \tau \tag{3.11b}$$

$$\sum_{j=0}^{\infty} p_j(-2y)p_j(\tilde{D}) \exp\left(\sum_{l=1}^{\infty} y_l D_l\right) \rho \cdot \widetilde{\tau} = \exp\left(\sum_{l=1}^{\infty} y_l D_l\right) \widetilde{\tau} \cdot \rho \tag{3.11c}$$

where $y = (y_1, y_2, \ldots)$ is an arbitrary parameter, $\tilde{D} = (D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \cdots)$, D_i is the Hirota's bilinear operator with respect to t_i and $p_j(t)(j = 1, 2, \ldots)$ is the Schur polynomial defined by

$$e^{\xi(t,\lambda)} = \sum_{j=0}^{\infty} p_j(t)\lambda^j. \tag{3.12}$$

If we let $k = 1$, then the simplest set of equations in (3.11) is

$$(D_1^2 + D_2)\tau \cdot \widetilde{\tau} = 2\rho\sigma \tag{3.13a}$$

$$(D_2 - D_1^2)\rho \cdot \widetilde{\tau} = 0 \tag{3.13b}$$

$$(D_2 + D_1^2)\sigma \cdot \tau = 0. \tag{3.13c}$$

By using the formula [6]

$$T(\widetilde{f}(x) - f(x)) = \mathrm{i}(\widetilde{f}(x) + f(x)) \tag{3.14}$$

where $\widetilde{f}(x) = f(x - 2\mathrm{i}\hbar)$, we can verify that equations (3.13) are the bilinear equations for equations (2.5) with q, r expressed by $\rho, \sigma, \tau, \widetilde{\tau}$, as in (3.8), with u_1 expressed as follows

$$u_1 = \widetilde{w}_1 - w_1 = \frac{\tau_x}{\tau} - \frac{\widetilde{\tau}_x}{\widetilde{\tau}}. \tag{3.15}$$

The above expression for u_1 is obtained from (2.4), (2.7) and (3.5).

4. Solutions for the hierarchy of equations (2.2)

We shall solve the bilinear equations in (3.10) to obtain solutions of the hierarchy of equations (2.2). We note that the function τ, which satisfies equations in (3.10), is also a τ-function for the KP hierarchy. So, we can make use of the known τ-functions of the KP hierarchy to obtain solutions of the bilinear equations in (3.10); in particular, we shall employ the τ-functions given by [1]. So, we need first to introduce some of the notation of [1] on the free fermion operators.

As in [1], we denote by ϕ_n, ϕ_n^* ($n \in \mathbb{Z}$) the free fermion operators which satisfy the following commutation relations:

$$[\phi_n, \phi_m]_+ = \phi_n\phi_m + \phi_m\phi_n = 0 \tag{4.1a}$$

$$[\phi_n^*, \phi_m^*]_+ = 0 \qquad [\phi_n, \phi_m^*]_+ = \delta_{n,m}. \tag{4.1b}$$

The non-commutative algebra $\mathbb{A}$ over $\mathbb{C}$ is then generated by $\phi_n, \phi_n^* (n \in \mathbb{Z})$ and 1. Define

$$G(V, V^*) = \{g \in \mathbb{A} | \exists g^{-1}, gVg^{-1} = V, gV^*g^{-1} = V^*\} \tag{4.2}$$

where $V = \oplus_{n\in\mathbb{Z}}\mathbb{C}\phi_n$, $V^* = \oplus_{n\in\mathbb{Z}}\mathbb{C}\phi_n^*$. For any $g \in G(V, V^*)$, define the following function:

$$\tau(t, g) = \langle \mathrm{vac}|e^{H(t)}g|\mathrm{vac}\rangle \tag{4.3}$$

where

$$H = \sum_{n\in\mathbb{Z}} \sum_{i=1}^{\infty} t_i \phi_n \phi_{n+i}^*$$

and the vaccum states $|\mathrm{vac}\rangle$, $\langle\mathrm{vac}|$ satisfy the following conditions:

$$\phi_n|\mathrm{vac}\rangle = \langle\mathrm{vac}|\phi_n^* = 0 \qquad (n < 0) \tag{4.4a}$$

$$\phi_n^*|\mathrm{vac}\rangle = \langle\mathrm{vac}|\phi_n = 0 \qquad (n \geqslant 0) \tag{4.4b}$$

$$\langle\mathrm{vac}|\mathrm{vac}\rangle = 1. \tag{4.5}$$

Then $\tau(t, g)$ is a τ-function of the KP hierarchy [1].

We now have the following proposition.

Proposition 4.1. Let $g \in G(V, V^*)$ and $g^{-1}\Lambda_m g = c_{ij}^{(m)}\phi_i\phi_j^*$, where $\Lambda_l = \sum_{n\in\mathbb{Z}} \phi_n\phi_{n+l}^*$ for $l \geqslant 1$. If $c_{ij}^{(m)}$ for $i \geqslant 0$, $j < 0$ satisfies the condition

$$\sum_{m=0}^{\infty} \frac{(2\mathrm{i}\hbar)^m}{m!} c_{ij}^{(m+k)} = d_i e_j \tag{4.6}$$

for certain positive integer k, then we have the following solution for the equations in (3.10):

$$\tau(t) = \tau(t, g) = \langle\mathrm{vac}|\mathrm{e}^{H(t)} g|\mathrm{vac}\rangle \qquad \widetilde{\tau}(t) = \langle\mathrm{vac}|\mathrm{e}^{\widetilde{H}(t)} g|\mathrm{vac}\rangle \tag{4.7a}$$

$$\rho(t) = \langle\mathrm{vac}|\phi_0^* \mathrm{e}^{\widetilde{H}(t)} g \sum_{i\geqslant 0} d_i \phi_i|\mathrm{vac}\rangle \tag{4.7b}$$

$$\sigma(t) = \langle\mathrm{vac}|\phi_{-1} \mathrm{e}^{H(t)} g \sum_{i<0} e_i \phi_i^*|\mathrm{vac}\rangle \tag{4.7c}$$

where $\widetilde{H}(t) = H(t_1 - 2\mathrm{i}\hbar, t_2, \ldots)$.

Proof. Let us first check that $\tau, \widetilde{\tau}, \rho, \sigma$, given by (4.7), satisfy equation (3.10*a*). From [1], we know that

$$\mathrm{e}^{\xi(t,\lambda)} \mathrm{e}^{-\xi(\tilde{\partial},\lambda^{-1})} \langle\mathrm{vac}|\mathrm{e}^{H(t)} = \langle\mathrm{vac}|\phi_0^* \mathrm{e}^{H(t)} \phi(\lambda) \tag{4.8a}$$

$$\lambda \mathrm{e}^{-\xi(t,\lambda)} \mathrm{e}^{\xi(\tilde{\partial},\lambda^{-1})} \langle\mathrm{vac}|\mathrm{e}^{H(t)} = \langle\mathrm{vac}|\phi_{-1} \mathrm{e}^{H(t)} \phi^*(\lambda) \tag{4.8b}$$

where

$$\tilde{\partial} = \left(\frac{\partial}{\partial t_1}, \frac{\partial}{2\partial t_2}, \frac{\partial}{3\partial t_3}, \ldots\right)$$

and

$$\phi(\lambda) = \sum_{i\in\mathbb{Z}} \phi_i \lambda^i \qquad \phi^*(\lambda) = \sum_{i\in\mathbb{Z}} \phi_i^* \lambda^{-i}. \tag{4.9}$$

Then, from (4.8a), we have

$$e^{\xi(t,\lambda)}e^{-\xi(\tilde{\partial},\lambda^{-1})}\langle\mathrm{vac}|e^{\widetilde{H}(t)} = e^{2i\hbar\lambda}\langle\mathrm{vac}|\phi_0^* e^{\widetilde{H}(t)}\phi(\lambda). \tag{4.10}$$

Since $g \in G(V, V^*)$, we have

$$\phi_l g = \sum_{i\in\mathbb{Z}} u_{i,l} g\phi_i \qquad \phi_l^* g = \sum_{i\in\mathbb{Z}} v_{l,i} g\phi_i^* \qquad l \in \mathbb{Z} \tag{4.11}$$

so

$$c_{ij}^{(m)} = \sum_{l\in\mathbb{Z}} u_{i,l} v_{m+l,j}. \tag{4.12}$$

By using (4.8b) and (4.10)–(4.12), we have

$$\begin{aligned}
&\mathrm{Res}_\lambda \left(\lambda^k \tilde{\tau}(t-\epsilon(\lambda))\tau(t'+\epsilon(\lambda))e^{\xi(t-t',\lambda)}\right) \\
&\quad = \mathrm{Res}_\lambda \left(\lambda^{k-1} e^{2i\hbar\lambda}\langle\mathrm{vac}|\phi_0^* e^{\widetilde{H}(t)}\phi(\lambda)g|\mathrm{vac}\rangle\langle\mathrm{vac}|\phi_{-1}e^{H(t')}\phi^*(\lambda)g|\mathrm{vac}\rangle\right) \\
&\quad = \sum_{n=0}^{\infty}\frac{(2i\hbar)^n}{n!}\sum_{l\in\mathbb{Z}}\langle\mathrm{vac}|\phi_0^* e^{\widetilde{H}(t)}\phi_l g|\mathrm{vac}\rangle\langle\mathrm{vac}|\phi_{-1}e^{H(t')}\phi^*_{n+l+k}g|\mathrm{vac}\rangle \\
&\quad = \sum_{n=0}^{\infty}\sum_{l\in\mathbb{Z}}\sum_{i\geqslant 0, j<0}\frac{(2i\hbar)^n}{n!}\langle\mathrm{vac}|\phi_0^* e^{\widetilde{H}(t)}g\phi_i|\mathrm{vac}\rangle u_{i,l}v_{n+k+l,j}\langle\mathrm{vac}|\phi_{-1}e^{H(t')}g\phi_j^*|\mathrm{vac}\rangle \\
&\quad = \sum_{n=0}^{\infty}\sum_{i\geqslant 0, j<0}\frac{(2i\hbar)^n}{n!}c_{ij}^{(n+k)}\langle\mathrm{vac}|\phi_0^* e^{\widetilde{H}(t)}g\phi_i|\mathrm{vac}\rangle\langle\mathrm{vac}|\phi_{-1}e^{H(t')}g\phi_j^*|\mathrm{vac}\rangle \\
&\quad = \langle\mathrm{vac}|\phi_0^* e^{\widetilde{H}(t)}g\sum_{i\geqslant 0}d_i\phi_i|\mathrm{vac}\rangle\langle\mathrm{vac}|\phi_{-1}e^{H(t')}g\sum_{j<0}e_j\phi_j^*|\mathrm{vac}\rangle \\
&\quad = \rho(t)\sigma(t').
\end{aligned}$$

Similarly, we can prove that $\tau, \tilde{\tau}, \rho, \sigma$ also satisfy equations (3.10b) and (3.10c) by using the following identities from [1]:

$$e^{-\xi(t,\lambda)}e^{\xi(\tilde{\partial},\lambda^{-1})}\langle\mathrm{vac}|\phi_0^* e^{H(t)} = \langle\mathrm{vac}|e^{H(t)}\phi^*(\lambda) \tag{4.13a}$$

$$\lambda^{-1}e^{\xi(t,\lambda)}e^{-\xi(\tilde{\partial},\lambda^{-1})}\langle\mathrm{vac}|\phi_{-1}e^{H(t)} = \langle\mathrm{vac}|e^{H(t)}\phi(\lambda) \tag{4.13b}$$

the identities in (4.8) and the fact that

$$\sum_{n\in\mathbb{Z}} u_{l,n}v_{n,m} = \delta_{l,m}.$$

We have thus proved the proposition. □

By using proposition 4.1, we can obtain rational and soliton solutions for the hierarchy of equations (2.2). For example, let us take $g = e^{a\phi_i^*\phi_j}$ and assume that $i < -k, 0 \leqslant j < k$, where k is a positive integer. Then, for any integer $m \geqslant k$, we have

$$g^{-1}\Lambda_m g = \Lambda_m + a(\phi_j\phi_{i+m}^* - \phi_{j-m}\phi_i^*) - a^2\delta_{j,i+m}\phi_j\phi_i^*. \tag{4.14}$$

It is now easy to see that g satisfies the requirement of proposition 4.1. Let us denote by n_1 the largest integer such that $i + k + n_1 < 0$, and $n_2 = j - k - i$, then we can choose $d_j = 1$, $e_{i+k+n} = ((2i\hbar)^n/n!)a$ $(n = 0, 1, \ldots, n_1)$, $e_i = -a^2(2i\hbar)^{n_2}/n_2!$, and all other d_l, e_s with $l \geqslant 0$, $s < 0$ equal to zero. So, we obtain the following solution for the bilinear equations in (3.10):

$$\begin{aligned}\tau(t) &= \langle \mathrm{vac}|e^{H(t)}e^{a\phi_i^*\phi_j}|\mathrm{vac}\rangle = \langle \mathrm{vac}|e^{H(t)}(1 + a\phi_i^*\phi_j)|\mathrm{vac}\rangle \\ &= 1 + a\sum_{n\geqslant 0} p_{n-i}(-t)p_{j-n}(t) \end{aligned}\tag{4.15a}$$

$$\widetilde{\tau}(t) = \tau(t_1 - 2i\hbar, t_2, \ldots) \tag{4.15b}$$

$$\rho(t) = \langle \mathrm{vac}|\phi_0^* e^{\widetilde{H}(t)}e^{a\phi_i^*\phi_j}\phi_j|\mathrm{vac}\rangle = p_j(t_1 - 2i\hbar, t_2, \ldots) \tag{4.15c}$$

$$\begin{aligned}\sigma(t) &= \langle \mathrm{vac}|\phi_{-1}e^{H(t)}e^{a\phi_i^*\phi_j}\left(\sum_{n=0}^{n_1} a\frac{(2i\hbar)^n}{n!}\phi_{i+k+n}^* - a^2\frac{(2i\hbar)^{n_2}}{n_2!}\phi_i^*\right)|\mathrm{vac}\rangle \\ &= a\sum_{n=0}^{n_1}\frac{(2i\hbar)^n}{n!}X_{i+k+n+1}^*\tau(t) - a^2\frac{(2i\hbar)^{n_2}}{n_2!}X_{i+1}^*\tau(t)\end{aligned} \tag{4.15d}$$

where the operator X_l^* is defined as the coefficient of λ^{-l} when $e^{-\xi(t,\lambda)}e^{\xi(\tilde{\partial},\lambda^{-1})}$ is expanded in Laurent series in λ, more explicitly, X_l^* is given by [10]

$$X_l^* = \sum_{n\geqslant 0} p_{n-l}(-t)p_n(\tilde{\partial}). \tag{4.16}$$

In the second equality of (4.15*d*), we have used formula (4.8*b*). In the more special case when $k = 1$, $i = -2$, $j = 0$, we have, from (4.15),

$$\tau(t) = 1 + \tfrac{1}{2}at_1^2 - at_2 \qquad \widetilde{\tau}(t) = 1 + \tfrac{1}{2}a(t_1 - 2i\hbar)^2 - at_2 \tag{4.17a}$$

$$\rho(t) = 1 \qquad \sigma(t) = a - a^2 t_2 - \tfrac{1}{2}a^2t_1^2 + 2i\hbar a^2 t_1. \tag{4.17b}$$

The functions $\tau, \widetilde{\tau}, \rho, \sigma$, defined by (4.15), give a rational solution for the hierarchy of equations (2.2).

As a second example, let us take

$$g = \exp\left(\sum_{i,j=1}^{N} a_{ij}\phi(p_i)\phi^*(q_j)\right) \tag{4.18}$$

where $\phi(p_i)$ and $\phi^*(q_j)$ are defined as in (4.9) and the constants $a_{ij} (i, j = 1, 2, \ldots, N)$ have the form $\prod_{i,j=1}^{N}(p_i - q_j)b_{ij}$. Then, by a direct calculation, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(2i\hbar)^n}{n!}c_{ij}^{(n+k)} &= \sum_{n=0}^{\infty}\sum_{l,m=1}^{N}\frac{(2i\hbar)^n}{n!}a_{lm}p_l^i q_m^{-j}(p_l^{n+k} - q_m^{n+k}) \\ &= \sum_{l,m}^{N} a_{lm}p_l^i q_m^{-j}(p_l^k e^{2i\hbar p_l} - q_m^k e^{2i\hbar q_m}).\end{aligned} \tag{4.19}$$

We further assume that $p_l^k e^{2i\hbar p_l} - q_m^k e^{2i\hbar q_m} \neq 0$. Then, if we choose

$$a_{lm} = \frac{d_l e_m}{p_l^k e^{2i\hbar p_l} - q_j^k e^{2i\hbar q_m}} \tag{4.20}$$

we know that g satisfies the requirement of proposition 4.1 and we obtain the following solution for the bilinear equations in (3.10):

$$\tau(t) = \langle \mathrm{vac}|e^{H(t)} \exp\left(\sum_{i,j=1}^{N} a_{ij}\phi(p_i)\phi^*(q_j)\right)|\mathrm{vac}\rangle \qquad \tilde{\tau}(t) = \tau(t_1 - 2i\hbar, t_2, \ldots) \tag{4.21a}$$

$$\rho(t) = \langle \mathrm{vac}|\phi_0^* e^{\tilde{H}(t)} \exp\left(\sum_{i,j=1}^{N} a_{ij}\phi(p_i)\phi^*(q_j)\right)\sum_{i=1}^{N} d_i\phi(p_i)|\mathrm{vac}\rangle \tag{4.21b}$$

$$\sigma(t) = \langle \mathrm{vac}|\phi_{-1} e^{H(t)} \exp\left(\sum_{i,j=1}^{N} a_{ij}\phi(p_i)\phi^*(q_j)\right)\sum_{i=1}^{N} e_i\phi^*(q_i)|\mathrm{vac}\rangle \tag{4.21c}$$

then the functions τ, $\tilde{\tau}$, ρ, σ, given by (4.21), lead to the N-soliton solution for the hierarchy of equations (2.2). As an explicit example, let us take $N = 2$, then we have

$$\tau(t) = 1 + \sum_{i,j=1}^{2} \frac{a_{ij}q_j}{p_i - q_j} e^{\xi(t,p_i)-\xi(t,q_j)} + \frac{cq_1q_2(p_1-p_2)(q_1-q_2)}{(p_1-q_1)(p_2-q_2)(p_1-q_2)(q_1-p_2)} e^{\xi(t,p_1)+\xi(t,p_2)-\xi(t,q_1)-\xi(t,q_2)} \tag{4.22a}$$

$$\tilde{\tau}(t) = \tau(t_1 - 2i\hbar, t_2, \ldots) \tag{4.22b}$$

$$\rho(t) = \sum_{j=1}^{2} d_j e^{-2i\hbar p_j} e^{\xi(t,p_j)} + \sum_{j=1}^{2} f_j \frac{q_j(p_1-p_2)(d_1a_{2j} - d_2a_{1j})}{(p_1-q_j)(p_2-q_j)} e^{\xi(t,p_1)+\xi(t,p_2)-\xi(t,q_j)} \tag{4.22c}$$

$$\sigma(t) = \sum_{j=1}^{2} e_j q_j e^{-\xi(t,q_j)} + \sum_{j=1}^{2} \frac{q_1q_2(q_2-q_1)(e_1a_{j2} - e_2a_{j1})}{(p_j-q_2)(p_j-q_1)} e^{\xi(t,p_j)-\xi(t,q_1)-\xi(t,q_2)} \tag{4.22d}$$

where $c = a_{11}a_{22} - a_{12}a_{21}$, $f_j = e^{-2i\hbar(p_1+p_2-q_j)}$ and a_{ij} is defined by (4.20).

From (4.19) and the proof of proposition 4.1, we know that if $p_i^k e^{2i\hbar p_i} = q_i^k e^{2i\hbar q_i}$ and $a_{ij} = 0$ for $i \neq j$, then g defined by (4.18) leads to the solution of equations (3.10) with $\rho = \sigma = 0$, which are simply the bilinear equations for the ILW$_k$ hierarchy, and this result coincides with the one given in [5].

5. Conclusion

We present a class of integro-differential equations which generalizes the ILW$_k$ hierarchy, and includes the coupling of the ILW equation and the time-dependent Schrödinger equations as the simplest example. We have shown the generalized Lax pairs and the bilinear representations for these integro-differential equations, and by making use of the bilinear representations, we have obtained their rational and N-soliton solutions. All the above properties of these integro-differential evolution equations are important and are commonly shared by integrable nonlinear evolution equations. Therefore, we expect that these integro-differential equations are integrable. We end this paper with the following two remarks.

Remark 1. Since having Hamiltonian structure is one of the basic features for integrable non-linear evolution equations, it is interesting to consider the Hamiltonian structure for the hierarchy of equations (2.2). For example, motivated by [7,9], we know that equations in (2.5) have the following Hamiltonian formalism:

$$q_{t_2} = \{\mathcal{H}, q\} \qquad u_{1,t_2} = \{\mathcal{H}, u_1\} \qquad r_{t_2} = \{\mathcal{H}, r\} \tag{5.1}$$

where the Hamiltonian $\mathcal{H} = \int_{-\infty}^{\infty}(\frac{1}{3}u_1^3 + \frac{1}{2}u_1 T u_{1,x} - u_1 qr + q_x r)\,\mathrm{d}x$ and, for two functionals $I = \int_{-\infty}^{\infty} f\,\mathrm{d}x$ and $J = \int_{-\infty}^{\infty} g\,\mathrm{d}x$, the Poisson bracket $\{I, J\}$ is defined as

$$\begin{aligned}\{I, J\} = \int_{-\infty}^{\infty} \Bigg\{ & \frac{\delta g}{\delta u_1}\left[-\left(\frac{\delta f}{\delta u_1}\right)_x + r\frac{\delta f}{\delta r} - q\frac{\delta f}{\delta q}\right] + \frac{\delta g}{\delta r}\left[\left(\frac{\delta f}{\delta q}\right)_x - u_1\frac{\delta f}{\delta q} - r\frac{\delta f}{\delta u_1}\right] \\ & + \frac{\delta g}{\delta q}\left[\left(\frac{\delta f}{\delta r}\right)_x + u\frac{\delta f}{\delta r} + q\frac{\delta f}{\delta u_1}\right] \\ & + \left(q\frac{\delta g}{\delta q} - r\frac{\delta g}{\delta r}\right)\left[\left(\partial^{-1} r\frac{\delta f}{\delta r}\right) - \left(\partial^{-1} q\frac{\delta f}{\delta q}\right)\right]\Bigg\}\,\mathrm{d}x.\end{aligned} \tag{5.2}$$

This Poisson bracket is simply the Poisson bracket $\{\,,\}^{(0)}$ constructed to deduce the Hamiltonian structure of the 1-constrained KP hierarchy in [7], and it is reduced from the second Poisson bracket for the KP hierarchy. The Hamiltonian structure for the whole hierarchy of equations (2.2) will be considered elsewhere.

Remark 2. Constraint condition (2.1) can be further generalized to the following condition:

$$(\widetilde{P}\partial^k P^{-1})_- = \sum_{i=1}^{N} q_i \partial^{-1} r_i \tag{5.3}$$

and we obtain the following hierarchy of integro-differential equations:

$$L_{t_n} = \widetilde{B}_n L - L B_n \tag{5.4a}$$

$$q_{it_n} = \widetilde{B}_n q_i \tag{5.4b}$$

$$r_{it_n} = -B_n^* r_i \qquad i = 1, 2, \ldots, N \qquad n = 1, 2, \ldots. \tag{5.4c}$$

The above considerations for the hierarchy of equations (2.2) can also be applied to the hierarchy of equations (5.4).

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